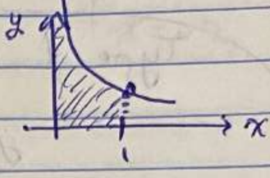
Type II improper integrals (vertical Asymptote)

example $\int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_{x=a}^{1}$



$= 2(1-\sqrt{a})$

in the limit $\lim_{a\to 0} \int_a^1 \frac{dx}{\sqrt{x}} = 2$

defn.

1) if $f(x)$ continuous on $(a,b]$, discontinuous at a then define

1) $\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$

2) if $f(x)$ continuous on $[a,b)$, discontinuous at b then define

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$

3) if $f(x)$ discontinuous at c, but continuous on $[a,c) \cup (c,b]$, then define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)\,dx$$

both integrals on RHS are defined as above example $\int_0^1 \frac{1}{1-x}\,dx$ singularity at $x=1$

$$\int_0^b \frac{1}{1-x}\,dx = \int \frac{-1}{u}\,du = -\ln|u|$$

$$= -\ln|1-x|\Big|_{x=0}^{b}$$

$$= -\ln|1-b| + 0$$

in the limit

$$\lim_{b\to 1}\left(-\ln|1-b|\right) = +\infty$$

this is <u>divergent</u>.

motivates discussion of

$$\int_0^1 \frac{1}{x^p}\,dx$$

$p<1$	convergent
$p=1$	divergent
$p>1$	divergent

Tests for convergence and divergence

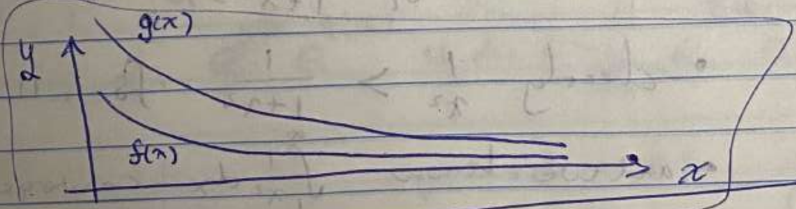
<u>Direct Comparison test</u>

let f and g be continuous on $[a,\infty)$ with $f(x) \le g(x)$ for all $x \ge a$

then

If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ also converges

If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ also diverges



example $\int_1^\infty e^{-x^2}\,dx$ compare $0 \le e^{-x^2} \le e^{-x}$

on the interval $x\in[1,\infty)$

example $\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$ compare to $\frac{1}{x^2}$

example $\int_1^{\infty} \frac{1}{\sqrt{x^2+1}}\,dx$ compare to $\frac{1}{x}$

example $\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$ compare to $\frac{1}{\sqrt{x}}$

Limit comparison test

if positive fncs f and g are continuous on $[a, \infty)$ and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$$

then

$$\int_a^{\infty} f(x)\,dx = \int_a^{\infty} g(x)\,dx$$

either both converge or both diverge

example show $\int_1^{\infty} \frac{dx}{1+x^2}$ converges by comparison with $\int_1^{\infty} \frac{1}{x^2}\,dx$

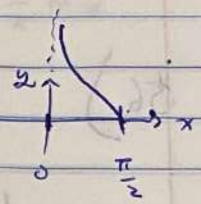
- clearly $\frac{1}{x^2} > \frac{1}{1+x^2}$ for all $x \in [1, \infty)$
- and we know $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges
- pose $\lim_{x\to\infty} \left(\frac{1/x^2}{\frac{1}{1+x^2}}\right) = \lim_{x\to\infty} \frac{1+x^2}{x^2}$

$$= \lim_{x\to\infty} \left(\frac{(1/x^2)+1}{1}\right) = 1 \quad \text{finite and nonzero}$$

- therefore $\int_1^{\infty} \frac{1}{1+x^2}\,dx$ converges

Test for convergence

40) $\int_0^{\pi/2} \cot\theta \, d\theta$



$\int_0^{\pi/2} \frac{\cos\theta}{\sin\theta} d\theta$ $\quad u = \sin\theta \quad du = \cos\theta$

θ	u
0	0
$\frac{\pi}{2}$	1

$= \int_0^1 \frac{1}{u} du$

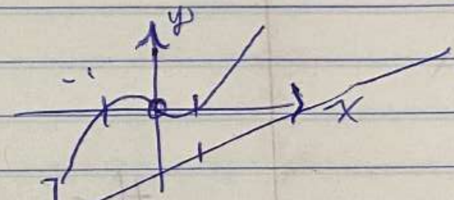
$= \ln|u| \Big|_{u=0}^{1}$

or

$= \lim_{\varepsilon \to 0} \left(\ln|u| \Big|_{u=\varepsilon}^{1} \right) = \lim_{\varepsilon \to 0} \left(\ln(1) - \ln(\varepsilon) \right)$

$= -\infty$ diverges

50) $\int_{-1}^{1} x \ln|x| \, dx$



$= \lim_{c \to 0^+} \left[\int_{-1}^{-c} x \ln|x| \, dx + \int_c^1 x \ln|x| \right] dx$

Let $u = \ln|x|$ then $du = \frac{1}{x} dx$

bounds

x	u
-1	0
$-c$	$\ln\lvert c\rvert$
$+c$	$\ln\lvert c\rvert$
1	0

$= \lim_{c \to 0^+} \left[\int_{u=0}^{\ln c} |x|^2 u \, du + \int_{\ln c}^{0} |x|^2 u \, du \right]$

$= \lim_{c \to 0^+} \left[\int_0^{\ln c} e^{2|u|} u \, du + \int_{\ln c}^{0} e^{2u} u \, du \right]$

52) $\int_4^\infty \frac{dx}{\sqrt{x}-1}$ $\quad$ guess diverges

Compare with $\int_4^\infty \frac{1}{\sqrt{x}} dx$

$\frac{1}{\sqrt{x}} < \frac{1}{\sqrt{x}-1}$ and $\int_4^\infty \frac{1}{\sqrt{x}} dx = 2\sqrt{x} \Big|_4^\infty$

which diverges

So $\int_4^\infty \frac{dx}{\sqrt{x}-1}$ diverges by direct comparison

can also use limit comparison

56) $\int_2^\infty \frac{dx}{\sqrt{e^{2x}-1}}$ compare with $\int_2^\infty \frac{1}{e^x}\,dx$

~~direct comparison~~ ? we know $\int_2^\infty e^{-x}\,dx = -e^{-x}\Big|_{x=2}^{\infty}$

$= +e^{-2}$

converges!

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{1/\sqrt{e^{2x}-1}}{1/e^{x}}$$

$$= \lim_{x\to\infty}\frac{e^{x}}{\sqrt{e^{2x}-1}}\left(\frac{e^{-x}}{e^{-x}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{1}{\sqrt{1-e^{-2x}}}\right)$$

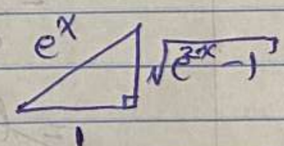
$= 1$ finite and nonzero

thus converges by limit comparison

also direct comparison?

$\frac{1}{\sqrt{e^{2x}-1}} > \frac{1}{e^{x}}$ on $x\in[2,\infty)$

it will work



$e^{x} > \sqrt{e^{2x}-1}$

8.8 Improper integrals

Type 1 improper integrals

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

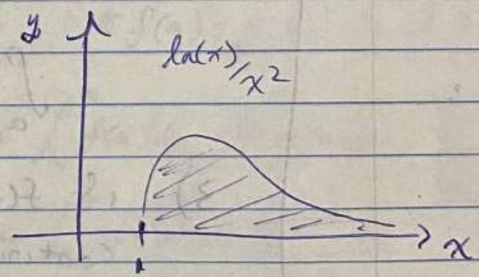
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

c can be any real number

If $\lim_{s\to\infty} \int_a^s f(x)\,dx$ exists and is finite the improper integral **converges**. if not the improper integral **diverges**.

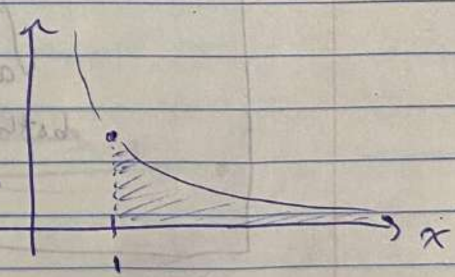
example $\int_1^{\infty} \frac{\ln(x)}{x^2}\,dx$

$\int_1^R \frac{\ln(x)}{x^2}\,dx$ by parts



example $\int_1^{\infty} \frac{1}{x^p}\,dx$

very important example



$$\int_1^R \frac{1}{x^p}\,dx = \left.\begin{cases} \ln(|x|) & \text{if } p=1 \\ \frac{-1}{(p-1)}\frac{1}{x^{p-1}} & \text{if } p>1 \end{cases}\right|_{x=1}^{R}$$

case $p=1$, $\int_1^R \frac{1}{x}\,dx = \log R - \log 1$
grows without bound for $R\to\infty$, **Diverges**

case $p>1$ $\int_1^R \frac{1}{x^p}\,dx = \frac{-1}{(p-1)}\left(\frac{1}{R^{p-1}} - \frac{1}{1}\right)$
approaches $\frac{1}{p-1}$ for $R\to\infty$
converges